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On a Problem of Sakai in Unbounded Derivations

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Let A_θ be the irrational rotation algebra. Then there exist non approximately bounded pregenerators δ_i $(i = 1, 2)$ of A_θ with the same domain such that given a $*$-derivation δ of A_θ with $D(\delta) = D(\delta_i)$, there exist $(k, l) \in \mathbb{R}^2$ and an approximately bounded $*$-derivation δ_3 of A_θ such that $\delta = k\delta_1 + l\delta_2 + \delta_3$, which can be considered as a solution of a problem of Sakai for two dimensional space quantizations.

1. INTRODUCTION

As a quantization of spaces, Sakai [10] raised the following problem: Can one find a simple C^*-algebra A and a non approximately bounded pregenerator δ_0 of A such that given a $*$-derivation δ of A with the same domain as δ_0, there exist a $k \in \mathbb{R}$ and an approximately bounded $*$-derivation δ_1 of A such that $\delta = k\delta_0 + \delta_1$?

In this paper, we answer the above problem in the two dimensional case. Actually, we show the following statement.

THEOREM. *Let $A = C(T)$ be the C^*-algebra of all continuous functions on the one dimensional torus T, and $\beta_t = \exp t\delta_0$ be the shift action of T on $C(T)$. Suppose $G \neq \{e\}$ is a discrete abelian group and α is a faithful homomorphism of G on A commuting with β, then given a $*$-derivation δ of $A \times_\alpha G$ with the same domain as $\tilde{\delta}_0$, there exist a $k \in \mathbb{R}$, a pregenerator δ_1, an approximately bounded $*$-derivation δ_2, and a $*$-derivation δ_x determined by an element x in $Z^1_\alpha(G, A)$ such that* (i) *$D(\delta_J)$ $(J = 1, 2) = D(\tilde{\delta}_0)$, $\tilde{\delta}_0$ commutes with δ_1, and* (ii) *$\delta = k\tilde{\delta}_0 + \delta_1 + \delta_2 + \delta_x$, where $\tilde{\delta}_0(x)(g) = \delta_0(x(g))$ for x in $D(\delta_0) \odot_\alpha G$, the $*$-algebra of all $D(\delta_0)$-valued functions on G with finite support, and $\delta_x(a\lambda(g)) = ax_g\lambda(g)$ $(a \in D(\delta_0))$, for $x \in Z^1_\alpha(G, A)$, the set of all α-one cocycles of G to A vanishing at e.*

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COROLLARY. *There exist a simple C^*-algebra A with unit and two non approximately bounded pregenerators δ_1, δ_2 of A with the same domain such that given a $*$-derivation δ of A with $D(\delta) = D(\delta_j)$ $(j = 1, 2)$, there are two real numbers k, l depending only on δ, and an approximately bounded $*$-derivation δ_3 of A such that* (i) $D(\delta_3) = D(\delta)$ *and* (ii) $\delta = k\delta_1 + l\delta_2 + \delta_3$.

General theory of unbounded derivations and crossed products are referred to [9] and [8]. We also mention that our groups are assumed to be abelian throughout the paper.

2. Several Lemmas

In order to show our Theorem, we prepare several lemmas.

LEMMA 1. *Let (A, G, α) and (A, H, β) be C^*-dynamical systems where α and β commute. Then we have that $(A \times_\alpha G)^{\tilde{\beta}} = A^\beta \times_\alpha G$, where $\tilde{\beta}_h(x)(g) = \beta_h(x(g))$ for $x \in L^1(G; A)$.*

Proof. By definition, $A^\beta \times_\alpha G$ is contained in $(A \times_\alpha G)^{\tilde{\beta}}$. If the inclusion is proper, it follows from [8] that $(A^\beta \times_\alpha G) \times_{\hat{\alpha}} \hat{G}$ is properly contained in $(A \times_\alpha G)^{\tilde{\beta}} \times_{\hat{\alpha}} \hat{G}$ since both are $\hat{\alpha}$-invariant. The latter is contained in $(A \times_\alpha G \times_{\hat{\alpha}} \hat{G})^{\tilde{\tilde{\beta}}}$. By duality [11], we have a contradiction. Q.E.D.

LEMMA 2. *Let α be an action of a discrete group G on a unital C^*-algebra A. Let δ_0 be a quasi well-behaved closed $*$-derivation of A commuting with α. Suppose δ is a $*$-derivation of $A \times_\alpha G$ such that* (i) $D(\delta) = D(\tilde{\delta}_0)$ *and* (ii) *δ commutes with $\hat{\alpha}$, then δ is closable.*

Proof. Let $x \in D(\delta)$ such that $x_n \to 0$, $\delta(x_n) \to y \in A \times_\alpha G$. Since $x_n = \sum_g a_g^n \lambda(g)$ $(a_g \in D(\delta_0))$, $\delta(x_n) = \sum_g \{\delta(a_g^n)\lambda(g) + a_g^n \delta(\lambda(g))\}$. Let Φ be the canonical projection of norm one from $A \times_\alpha G$ onto A. By (ii), $\delta(a_g^n) \in A$ and $\Phi(\delta(\lambda(g))\lambda(h)^*) = \delta(\lambda(g))\lambda(g)^*$ if $g = h$, $= 0$ otherwise. Let $y = \sum_g y_g \lambda(g)$ be the Fourier expansion of y $(y_g \in A)$. Then $a_g^n \to 0$ and $\delta(a_g^n) + a_g^n \delta(\lambda(g))\lambda(g)^* \to y_g$ $(g \in G)$. So $\delta(a_g^n) \to y_g$. Since $D(\delta|_A) = D(\delta_0)$ and δ_0 is quasi well-behaved closed, $\delta|_A$ is closable by [2]. Hence $y_g = 0$ $(g \in G)$ which implies $y = 0$. Q.E.D.

LEMMA 3. *Let (A, G, α) be as in Lemma 2, and δ_0 be a closed $*$-derivation of A commuting with α. Suppose δ is a $*$-derivation of $A \times_\alpha G$ such that $D(\delta) = D(\tilde{\delta}_0)$, then δ is relatively bounded on $D(\delta_0)$ with respect to δ_0.*

Proof. It suffices to show that δ is closable from the Banach $*$-algebra $D(\delta_0)$ into $A \times_\alpha G$. Let $a_n \in D(\delta_0)$ so that $a_n \to 0$ in $D(\delta_0)$, $\delta(a_n) \to x \in$

$A \times_\alpha G$. Let I be the set of all $b \in D(\delta_0)$ such that $a \to \delta(ab)$ is continuous from $D(\delta_0)$ into $A \times_\alpha G$. Since $\delta(a\lambda(g)b) = \delta(\lambda(g))\,\alpha_g^{-1}(a)b + \lambda(g)\,\delta(\alpha_g^{-1}(a)b)$, and δ_0 commutes with α, we have $x\lambda(g)b = 0$ for all $g \in G$ and $b \in I$. Then $\Phi(x\lambda(g))b = 0$, where Φ is as Lemma 2. So $\Phi(x\lambda(g))$ is in the left annihilator $L(I)$ of I. Since I is a two-sided ideal of $D(\delta_0)$, it follows from the same way as [6] that $L(I) = 0$. Thus $\Phi(x\lambda(g)) = 0$ for all $g \in G$. Let $x = \sum x_g \lambda(g)$ be the Fourier expansion of x. Then $x_g = 0$. So $x = 0$.

Q.E.D.

Remark 1. Since $\tilde{\delta}_0$ is preclosed, one could not directly apply Longo's result. However, the crucial part of the above proof is due to him.

The next lemma is quite useful to show our main statement though the proof itself is elementary:

LEMMA 4. *Let (A, G, α), $\beta_t = \exp t\delta_0$, and δ be supposed as in Lemma 3. Suppose β is periodic, then there exist derivations $\hat{\rho}_g, \tilde{\rho}_n$ $(g \in G, n \in \mathbb{Z})$ of $A \times_\alpha G$ such that* (i) $D(\hat{\rho}_g) = D(\tilde{\rho}_n) = D(\delta)$, (ii) $\hat{\rho}_g = \int_{\hat{G}} \overline{\langle g, p\rangle}\, \hat{\alpha}_p \circ \delta \circ \hat{\alpha}_{-p}\, dp$, $\tilde{\rho}_n = \int_T e^{-int} \tilde{\beta}_t \circ \delta \circ \tilde{\beta}_{-t}\, dt$, (iii) $(\hat{\rho}_g)\tilde{}_0 = (\tilde{\rho}_0)\hat{}_g$ *commute with* $\tilde{\beta}$, $(\tilde{\rho}_n)\hat{}_e = (\hat{\rho}_e)\tilde{}_n$ *commute with* $\hat{\alpha}$.

Proof. It suffices to show that $t \to \delta \circ \tilde{\beta}_t(x)$ and $p \to \delta \circ \hat{\alpha}_p(x)$ are continuous for all $x \in D(\delta)$. Since $p \to \delta \circ \hat{\alpha}_p(a\lambda(g))$ is continuous, we only check that $t \to \delta \circ \tilde{\beta}_t(a\lambda(g))$ is continuous for $a \in D(\delta_0)$ and $g \in G$. By Lemma 3, there is a positive number K such that $\|\delta(a)\| \leqslant K\{\|a\| + \|\delta_0(a)\|\}$ for every a in $D(\delta_0)$. So we have that $\|\delta \circ \beta_t(a) - \delta \circ \beta_s(a)\| \leqslant M\{\|\beta_t(a) - \beta_s(a)\| + \|\beta_t \circ \delta_0(a) - \beta_s \circ \delta_0(a)\|\}$. Q.E.D.

Remark 2. The derivation $(\tilde{\rho}_0)\hat{}_e$ defined above commutes with $\hat{\alpha}$ and $\tilde{\beta}$.

LEMMA 5. *Let (A, G, α) be a C^*-dynamical system where A is unital abelian and G is discrete. Let $\beta_t = \exp t\delta_0$ be a transitive action of T on A commuting with α. Suppose δ is a $*$-derivation of $A \times_\alpha G$ such that* (i) $D(\delta) = D(\tilde{\delta}_0)$ *and* (ii) *δ commutes with $\hat{\alpha}, \tilde{\beta}$, then there exist a $k \in \mathbb{R}$ and a pregenerator δ_1 of $A \times_\alpha G$ such that* (i) $D(\delta_1) = D(\tilde{\delta}_0)$, $\delta_1|_A = 0$ *and* (ii) $\delta = k\tilde{\delta}_0 + \delta_1$.

Proof. By Lemma 2, δ is closable. So we may assume that δ is closed. Let $x \in C^*(G)$. Then there is a net $\{x_l\}$ of $D(\delta)$ which converge to x. Let $y_l = \int_T \tilde{\beta}_t(x_l)\, dt \in (A \times_\alpha G)^{\tilde{\beta}}$. Since $(A \times_\alpha G)^{\tilde{\beta}} = C^*(G)$ by Lemma 1, $y_l \to x$. Since δ commutes $\tilde{\beta}$ and δ is closed, we have that $y_l \in D(\delta) \cap C^*(G)$ and $\delta(y_l) \in C^*(G)$. So $\delta|_{C^*(G)}$ is a closed $*$-derivation of $C^*(G)$. Let F be the Fourier isomorphism of $C^*(G)$ onto the C^*-algebra $C(\hat{G})$ of continuous functions on $\hat{G}$. Let $\hat{\delta} = F \circ \delta \circ F^{-1}$. Since δ commutes α and $F \circ \alpha \circ F^{-1}$ is the shift of $\hat{G}$ on $C(\hat{G})$, it follows from Goodman [5] and Nakazato [7] that there exists a one parameter subgroup (p_t) of $\hat{G}$ such that $\hat{\delta}(f)(p) =$

$\lim_{t\to 0} t^{-1}(f(p_t p) - f(p))$ for all f in $D(\tilde{\delta})$. Since $\langle g, \cdot\rangle \in D(\tilde{\delta})$, we have $\delta(\lambda(g)) = \partial(g)\lambda(g)$ for all g in G, where $\partial(g) = \lim_{t\to 0} t^{-1}(\langle g, p_t\rangle - 1)$. Let $\delta_1(a\lambda(g)) = \partial(g)\, a\lambda(g)$ for $a \in D(\delta_0)$ and $g \in G$. Then it is a pregenerator of $A \times_\alpha G$ such that $D(\delta_1) = D(\tilde{\delta}_0)$ and $\delta_1|_A = 0$. Since δ is a closed *-derivation of A with $D(\delta|_A) = D(\delta_0)$, it follows from Batty [1] that there exists an $f \in A$ such that $\delta|_A = f\delta_0$. Since $\delta|_A$ commutes with β and $\beta_t = \exp t\delta_0$, $\beta_t(f) = f$ for all $t \in T$. So $f = k1$ for some $k \in \mathbb{R}$. Therefore we have $\delta(a\lambda(g)) = k\delta_0(a)\lambda(g) + a\partial(g)\lambda(g) = (k\tilde{\delta}_0 + \delta_1)(a\lambda(g))$. Q.E.D.

Remark 3. The pregenerator δ_1 defined above would be written as $\delta_1 = r\delta'$ for some $r \in \mathbb{R}$, where δ' is independent of δ. Actually if $G = \mathbb{Z}$, we have $\delta'(a\lambda(n)) = ina\lambda(n)$ for $a \in D(\delta)$ and $n \in \mathbb{Z}$.

LEMMA 6. *Let (A, G, α) be a C^*-dynamical system where A is abelian and G is discrete. Let δ be a linear mapping of a *-subalgebra $D(\delta)$ of A into A such that $\delta(ab) = \delta(a)\alpha_g(b) + a\delta(b)$, $(a, b \in D(\delta))$ for a fixed $g \neq e$. Suppose there exists a unitary $u \in D(\delta)$ such that $1 \notin \mathrm{sp}(\alpha_g(u)u^*)$, then we have $\delta = a_g(\alpha_g - id)$ on $D(\delta) \cap C^*(u)$ for some $a_g \in A$.*

Proof. By assumption, we have $\delta(u^n) = \sum_{k=0}^{n-1} \alpha_g(u^k) u^{-k}\delta(u) u^{n-1}$. Since $1 \notin \mathrm{sp}(\alpha_g(u)u^*)$, we have that $\sum_{k=0}^{n-1} \alpha(u^k) u^{-k} = (\alpha_g(u^n) u^{-n} - 1)(\alpha_g(u) u^* - 1)^{-1}$. So $\delta(u^n) = \delta(u) u^*(\alpha_g(u) u^* - 1)^{-1}(\alpha_g - id)(u^n) = \delta(u)(\alpha_g(u) - u)^{-1}(\alpha_g - id)(u^n)$ for all $n \in \mathbb{Z}$ since $\delta(1) = 0$. Put $a_g = \delta(u)(\alpha_g(u) - u)^{-1}$. Since $a_g(\alpha_g - id)$ is bounded on A, the conclusion follows.

Q.E.D.

Remark 4. By the above lemma, there is no unbounded α_g-cocycle closed *-derivation if α_g has an eigenunitary generating A.

LEMMA 7. *Let $A = C(T)$, and $\beta_t = \exp t\delta$ be the action of T on A associated to the rotation by t. Suppose α is a faithful action of a discrete group G on A commuting with β, then given a *-derivation δ of $A \times_\alpha G$ such that* (i) *$D(\delta) = D(\tilde{\delta}_0)$ and* (ii) *δ commutes with $\tilde{\beta}$, there exist a k in $\mathbb{R}$, a pregenerator δ_1, and an approximately bounded *-derivation δ_2 of $A \times_\alpha G$ such that* (i) *$D(\delta_1) = D(\delta)$, $\delta_1|_A = 0$,* (ii) *$D(\delta_2) = D(\delta)$, $\delta_2(\lambda(g)) = 0$ for all g in G, and* (iii) *$\delta = k\tilde{\delta}_0 + \delta_1 + \delta_2$.*

Proof. Let u be a unitary such that $A = C^*(u)$, $\beta_t(u) = e^{it}u$, $1 \notin \mathrm{sp}(\alpha_g(u)u^*)$, $g \neq e$. Let $\hat{\rho}_g$ $(g \in G)$ be as in Lemma 4. By definition, we have $\delta = \sum_g \hat{\rho}_g$ on $D(\delta)$ since $\hat{\rho}_g(a) = \delta(a)(g)\lambda(g)$ and $\hat{\rho}_g(\lambda(h)) = \delta(\lambda(h))(g+h)\lambda(g+h)$ for the Fourier expansion of $\delta(a) = \sum_g \delta(a)(g)\lambda(g)$ and $\delta(\lambda(h)) = \sum_g \delta(\lambda(h))(g)\lambda(g)$. Since δ commutes with $\tilde{\beta}$, it follows from Lemma 5 that $\hat{\rho}_e = k\tilde{\delta}_0 + \delta_1$ on $D(\delta)$, where k, δ_1 are as in Lemma 5. Let $\delta_g(a) = \hat{\rho}_g(a)\lambda(g)^*$ for a in $D(\delta_0)$ $(g \neq e)$. Then δ_g satisfy the condition of Lemma 6. So by assumption we have $\delta_g = a_g(\alpha_g - id)$ on $D(\delta_0)$ for some a_g

in A. Since δ commutes with $\tilde{\beta}$ and α commutes with β, we have $a_g \in \mathbb{C}1$. Then $\hat{\rho}_g(a) = a_g(\alpha_g - id)(a)\,\lambda(g) = [a_g\lambda(g), a]$. Hence $\hat{\rho}_g(a\lambda(h)) = \hat{\rho}_g(a)\,\lambda(h) + a\hat{\rho}_g(\lambda(h)) = [a_g\lambda(g), a\lambda(h)] + a\hat{\rho}_g(\lambda(h))$. Since $\hat{\rho}_g - \mathrm{ad}(a_g\lambda(g))$ is a derivation on $D(\delta)$, we have $\hat{\rho}_g(\lambda(h)) = 0$ for h in G. In fact, since $\hat{\rho}_g(\lambda(h)) = \delta(\lambda(h))(g+h)\,\lambda(g+h)$, we have that $\alpha_h^{-1} \circ \delta(\lambda(h+k))(h+k+g)u = \alpha_h^{-1} \circ \delta(\lambda(h))(h+g)\,\alpha_g(u) + \delta(\lambda(k))(k+g)u$ for all h and k in G. Since $1 \in D(\delta)$, we have $\delta(1)(g) = 0$. So $\delta(\lambda(h))(h+g) = 0$ for all h in G or $\alpha_g(u) = u$. Since $1 \notin \mathrm{sp}(\alpha_g(u)\,u^*)$, we have $\delta(\lambda(h))(h+g) = 0$ for all $h \in G$. Consequently $\delta = k\tilde{\delta}_0 + \delta_1 + \sum_{g\neq e} \mathrm{ad}(a_g\lambda(g))$ on $D(\delta)$. Let $\delta_F = \mathrm{ad}(\sum_{g\in F} a_g\lambda(g))$ for a finite set F of $G - \{e\}$ with $F = -F$. Then δ_F are bounded *-derivations of $A \times_\alpha G$ such that $\delta_F(\lambda(h)) = 0$ and δ_F converge to δ_2 pointwisely on $D(\delta)$, where $\delta_2(a\lambda(h)) = \sum_{g\neq e}[a_g\lambda(g), a\lambda(h)]$ $(= (\delta - \hat{\rho}_e)(a)\,\lambda(h))$. Then $\delta = k\tilde{\delta}_0 + \delta_1 + \delta_2$ on $D(\delta)$ and $\delta_2(\lambda(g)) = 0$ for all g in G. Q.E.D.

Remark 5. In the case of discrete abelian groups, the Fourier expansion of any element of $A \times_\alpha G$ can be taken in the uniform sense. In fact, taking a net $\{f_l\}$ of positive definite functions on G with finite supports converging to 1, one can show that $\sum_g f_l(g)\,a_g\lambda(g)$ converge to $\sum_g a_g\lambda(g) \in A \times_\alpha G$ uniformly.

3. Proof of Theorem

Let u be a unitary as in Lemma 7. Since β commutes with α and β is ergodic, we have $\alpha_g(u)\,u^* \in \mathbb{C}1$. Since $A = C^*(u)$ and α is faithful, there are $c_g \neq 1$ $(g \neq e)$ such that $\alpha_g(u) = c_g u$. So $1 \notin \mathrm{sp}(\alpha_g(u)\,u^*)$ $(g \neq e)$. Let $\tilde{\rho}_n$ be as in Lemma 3. Since $\tilde{\rho}_0$ commutes with $\tilde{\beta}$, it follows from Lemma 7 that $\tilde{\rho}_0 = k'\tilde{\delta}_0 + \delta_1' + \delta_2'$, where δ_i' are as in Lemma 7. Since $\tilde{\beta}_t \circ \tilde{\rho}_n \circ \tilde{\beta}_t^{-1} = e^{int}\tilde{\rho}_n$ $(n \in Z)$, $\tilde{\beta}_t \circ \tilde{\rho}_n(\lambda(g)) = e^{itn}\tilde{\rho}_n(\lambda(g))$. Since $\beta_t(u^n) = e^{itn}u^n$, we have that $u^{-n}\tilde{\rho}_n(\lambda(g)) \in (A \times_\alpha G)^{\beta} = C^*(G)$. So there are $b(n,g) \in C^*(G)$ such that $\tilde{\rho}_n(\lambda(g)) = u^n b(n,g)$ $(n \in \mathbb{Z})$. Let $\delta(\lambda(g)) = \sum_h \delta(\lambda(g))(h)\,\lambda(h)$ be the Fourier expansion of $\delta(\lambda(g))$, and $b(n,g) = \sum_h b(n,g)(h)\,\lambda(h)$ be of $b(n,g)$. Then we have that $\delta(\lambda(g))(h) = a(0) + \sum_{n\neq 0} b(n,g)(h)\,u^n$ where $a(0)$ is the 0-component of the expansion of $\delta(\lambda(g))(h)$ in A since $A = C^*(u)$ and $\beta_t(u) = e^{it}u$. Since $\tilde{\rho}_0 = k'\tilde{\delta}_0 + \delta_1' + \delta_2'$, we have $\tilde{\rho}_0(\lambda(g)) = \partial(g)\,\lambda(g)$. By unicity, $\int_T \beta_t(\delta(\lambda(g))(h))\,dt = \partial(g)1$ $(g \neq h)$, $= 0$ (otherwise), which is nothing but $a\,(0)$. Therefore we deduce that $\delta(\lambda(g)) = \partial(g)\,\lambda(g) + \sum_h \sum_{n\neq 0} b(n,g)(h)\,u^n\lambda(h) = \partial(g)\,\lambda(g) + \sum_{n\neq 0} u^n b(n,g) = \partial(g)\,\lambda(g) + \sum_{n\neq 0}\tilde{\rho}_n(\lambda(g))$. Moreover, $\delta(a) = \sum_g \hat{\rho}_g(a)$ for all $a \in D(\delta_0)$. It follows from Lemma 6 that $\hat{\rho}_g(a) = f_g(\alpha_g - id)(a)\,\lambda(g)$ for some $f_g \in A$ $(g \neq e)$. So $\hat{\rho}_g(a) = [f_g\lambda(g), a]$ for all $a \in D(\delta_0)$. Since $\hat{\rho}_e$ commutes with $\hat{\alpha}$, we have $\hat{\rho}_e(a) \in A$ for all $a \in D(\delta_0)$. Since $(\hat{\rho}_e)_0^{\sim}$ commutes with $\hat{\alpha}$ and $\tilde{\beta}$, it follows from

Lemma 5 that $(\hat{\rho}_e)_0^{\sim} = k\tilde{\delta}_0 + \delta_1$, where k, δ_1 are as in Lemma 5. Then $\int_T e^{-it}\beta_t \circ \hat{\rho}_e(u)\,dt = k\delta_0(u)$. Since $\beta_t(u) = e^{it}u$, we have $\delta_0(u) = iu$. Let $\hat{\rho}_e(u) = \sum_n a_n u^n$ $(a \in \mathbb{C})$. Then $a_1 = ik$. Therefore $\hat{\rho}_e(u) = k\delta_0(u) + \sum_{n \neq 1} a_n u^n$. Since $\hat{\rho}_e$ is a *-derivation, we deduce that $\hat{\rho}_e(u^n) = n\hat{\rho}_e(u)\, u^{n-1} = kn\delta_0(u)\, u^{n-1} + \sum_{m \neq 1} na_m u^{m+n-1} = k\delta_0(u^n) + \sum_{m \neq 1} na_m u^{m+n-1}$. Hence $\hat{\rho}_e(u^n)\,\lambda(g) = k\tilde{\delta}_0(u\lambda(g)) + \sum_{m \neq 1} na_m u^{m+n-1}\lambda(g)$. Consequently, we have that $\delta(u^n\lambda(g)) = \delta(u^n)\,\lambda(g) + u^n\delta(\lambda(g)) = (k\tilde{\delta}_0 + \delta_1)(u^n\lambda(g)) + \sum_{h \neq e}[f_h\lambda(h), u^n]\,\lambda(g) + \sum_{m \neq 0} u^n\tilde{\rho}_m(\lambda(g)) + \sum_{m \neq 0} na_{m+1}u^{n+m}\lambda(g)$. Since $\delta - k\tilde{\delta}_0 - \delta_1$ is a *-derivation, so is $\sum_{h \neq e}[f_h\lambda(h), u^n]\,\lambda(g) + \sum_{m \neq 0} u^{n+m}\tilde{\rho}_m(\lambda(g)) + \sum_{m \neq 0} na_{m+1}u^{n+m}\lambda(g)$. Since $[f_h\lambda(h), u^n]\,\lambda(g) + u^n[f_h\lambda(h), \lambda(g)] = [f_h\lambda(h), u^n\lambda(g)]$, $u^n\lambda(g) \mapsto u^n(\sum_{m \neq 0} \tilde{\rho}_m(\lambda(g)) - \sum_{h \neq e}[f_h\lambda(h), \lambda(g)]) + \sum_{m \neq 0} na_{m+1}u^{m+n}\lambda(g)$ is a *-derivation. Let $a = \sum_{m \neq 0} a_{m+1}u^m \in A$. Conventionally, define $\sigma(\lambda(g)) = \sum_{m \neq 0}\tilde{\rho}_m(\lambda(g)) - \sum_{h \neq e}[f_h\lambda(h), \lambda(g)]$. We also put $\Delta(u^n\lambda(g)) = u^n\sigma(\lambda(g)) + nau^n\lambda(g)$. Since $\delta_0(u^n) = inu^n$, we see $nau^n\lambda(g) = (-i)\,a\delta_0(u^n\lambda(g))$. Now since $\Delta(u^n\lambda(g)\,u^m\lambda(h)) = \Delta(u^n\lambda(g))\,u^m\lambda(h) + u^n\lambda(g)\,\Delta(u^m\lambda(h))$, we can show that $u^{n+m}\sigma(\lambda(g))\;\lambda(h) - u^n\sigma(\lambda(g))\;\alpha_{-g}(u^m)\;\lambda(h) = m(\alpha_g(a) - a)\;u^{n+m}\lambda(g+h)$. Taking their norms of both hand sides, one has that $|m|\,\|\alpha_g(a) - a\| \leqslant 2\,\|\sigma(\lambda(g))\|$ for all m in $\mathbb{Z}$. Therefore $\alpha_g(a) = a$ for all g in G, which implies $a = c1$ for some c in $\mathbb{C}$. By definition of a, we see $a = 0$. So $u^m\sigma(\lambda(g)) = \sigma(\lambda(g))\,\alpha_{-g}(u^m)$ for all g in G and m in $\mathbb{Z}$. Since $\alpha_g(u) = c_g u$ and $c_g \neq 1$ $(g \neq e)$, we have that $\sigma(\lambda(g)) = \sigma(\lambda(g))(g)\,\lambda(g)$, where $\sigma(\lambda(g))(g)$ is the g-component of the Fourier expansion of $\sigma(\lambda(g))$. In fact, consider the Fourier coefficients $\sigma(\lambda(g))(k)$, $k \in G$, of $\sigma(\lambda(g))$. Since we know $\sigma(\lambda(g))\,\alpha_{-g}(u^m) = u^m\sigma(\lambda(g))$, we have $\sigma(\lambda(g))(k)\,\alpha_{n-g}(u^m) = u^m\sigma(\lambda(g))(k)$ for all k in G. Then $\sigma(\lambda(g))(k) = 0$ if $k \neq g$. Put $x_g = \sigma(\lambda(g))(g)$. Then $x_e = 0$. Since $\sigma(\lambda(g+h)) = \sigma(\lambda(g))\,\lambda(h) + \lambda(g)\,\sigma(\lambda(h))$, we have $x_{g+h} = x_g + \alpha_g(x_h)$ for all g, h in G. Namely, $x \in Z_\alpha^1(G, A)$, the set of all α–one cocycles of G into A vanishing at e. So we have that $\Delta(u^n\lambda(g)) = u^n x_g\lambda(g)$ for some x in $Z_\alpha^1(G, A)$. Therefore we have that $\delta(u^n\lambda(g)) = (c\tilde{\delta}_0 + \delta_1)(u^n\lambda(g)) + \sum_{h \neq e}[f_h\lambda(h), u^n\lambda(g)] + \delta_x(u^n\lambda(g))$ for all g in G and n in $\mathbb{Z}$, where $\delta_x(u^n\lambda(g)) = u^n x_g\lambda(g)$ for $x \in Z(G, A)$. Let $\delta_F(a\lambda(g)) = \sum_{h \in F}[f_h\lambda(h), a\lambda(g)]$ $(a \in D(\delta_0),\ g \in G)$ for a finite set F of $G - \{e\}$ with $F = -F$. Then δ_F is a bounded *-derivation for all F, and $\delta_F \to \delta_2$ pointwisely. Hence δ_2 is approximately bounded. This completes the proof. Q.E.D.

Proof of Corollary

Let A be the irrational rotation algebra, namely, the crossed product $C(T) \times_\alpha \mathbb{Z}$ of $C(T)$ by the action α of an irrational angle θ. Let $\beta_t = \exp t\delta_0$ be the shift action of T on $C(T)$. Then it suffices to show from Theorem and Remark 3 that given a x in $Z_\alpha^1(\mathbb{Z}, C(T))$, there exist a constant c and an approximately bounded derivation δ_2 of A such that $\delta_x = c\delta_1 + \delta_2$ on $D(\delta_0) \odot_\alpha \mathbb{Z}$, where $\hat{\alpha}_t = \exp t\delta_1$. In fact, let $x_n = \sum_{k \in \mathbb{Z}} c_{n,k}u^k \in C(T)$.

$c_{n,k} \in \mathbb{C}$. Since $x_{n+m} = x_n + \alpha_n(x_m)$ for all n, m in $\mathbb{Z}$, we have that $c_{n+m,k} = c_{n,k} + e^{ink\theta} c_{m,k}$ for all k, n, m in $\mathbb{Z}$. Then $c_{n,k} = (\sum_{j=0}^{n-1} e^{ijk\theta}) c_{1,k}$ $(n \geqslant 1)$. So $x_n = \sum_{k \neq 0} d_k(e^{ink\theta} - 1) u^k + c_{n,0}$, where $d_k = c_{1,k}(e^{ik\theta} - 1)^{-1}$ for $k \neq 1$. Therefore $\delta_x(u^m \lambda(n)) = u^m x_n \lambda(n) = \sum_{k \neq 0} d_k u^m (\alpha_n(u^k) - u^k) \lambda(n) + c_{n,0} u^m \lambda(n) = \sum_{k \neq 0} d_k [u^m \lambda(n), u^k] + c\delta_1(u^m \lambda(n))$ $(n \geqslant 1)$, where $c = (-i) c_{1,0}$. Since $d_k = c_{-1,k}(e^{-ik\theta} - 1)^{-1}$ for $k = 1$, we conclude that $\delta_x(u^m \lambda(n)) = \sum_{k \neq 0} \mathrm{ad}(-d_k u^k)(u^m \lambda(n)) + c\delta_1(u^m \lambda(n))$ for all m, n in $\mathbb{Z}$. This completes the proof. Q.E.D.

Remark 6. In the corollary, two pregenerators and are exclusively considered by Connes [3, 4].

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References

1. G. J. K. Batty, Derivations on compact spaces, preprint, 1978.
2. G. J. K. Batty, Small perturbations of C^*-dynamical systems, preprint, 1978.
3. A. Connes, C^*-algèbres et géométrie différentielle, preprint, 1980.
4. A. Connes, Feuilletages et algèbres d'opérateurs, preprint, 1980.
5. F. M. Goodman, Translation invariant closed *-derivations, preprint, 1980.
6. R. Longo, Automatic relative boundedness of derivations in C^*-algebras, *J. Funct. Anal.* **34** (1979), 21–28.
7. H. Nakazato, Closed *-derivations on compact groups, preprint, 1980.
8. G. K. Pedersen, "C^*-algebras and Their Automorphism Groups," Academic Press, New York, 1979.
9. S. Sakai, The theory of unbounded derivations in C^*-algebras, lecture notes, Copenhagen University and The University of Newcastle upon Tyne, 1977.
10. S. Sakai, Derivations in operator algebras, lecture notes, 1979.
11. H. Takai, On a duality for crossed products of C^*-algebras, *J. Funct. Anal.* **19** (1975), 25–39.